Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji

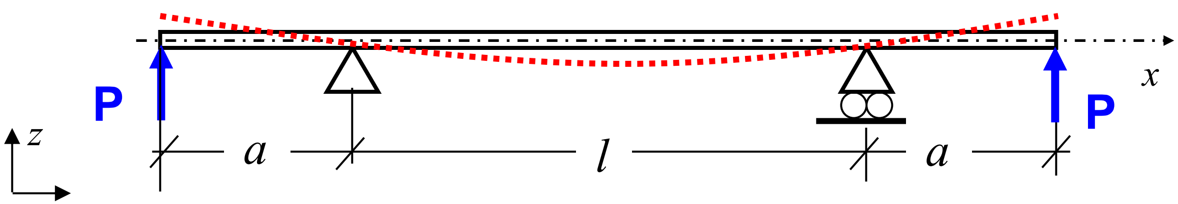
Wykład 10

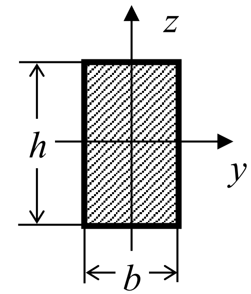
Pręty zginane – belki

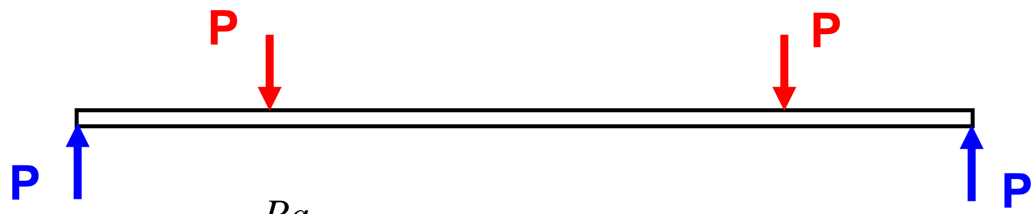
Wyznaczanie stanu naprężenia

Stan naprężenia w belce zginanej

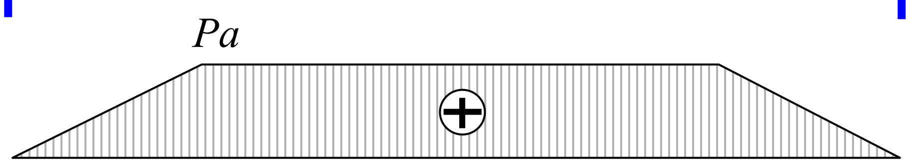
Czyste zginanie belki pryzmatycznej



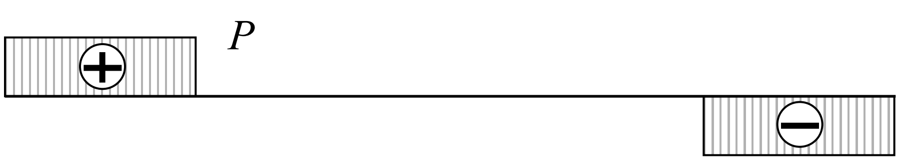




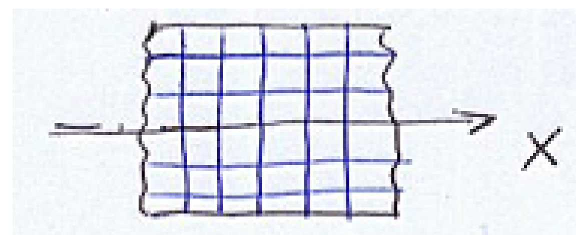
Mg(x)



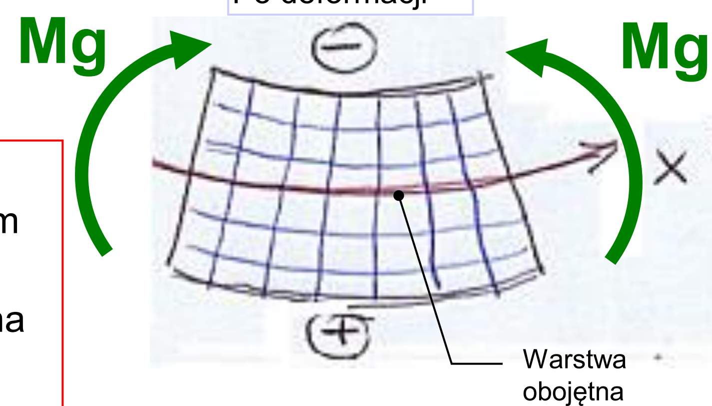
T(x)



Przed deformacją



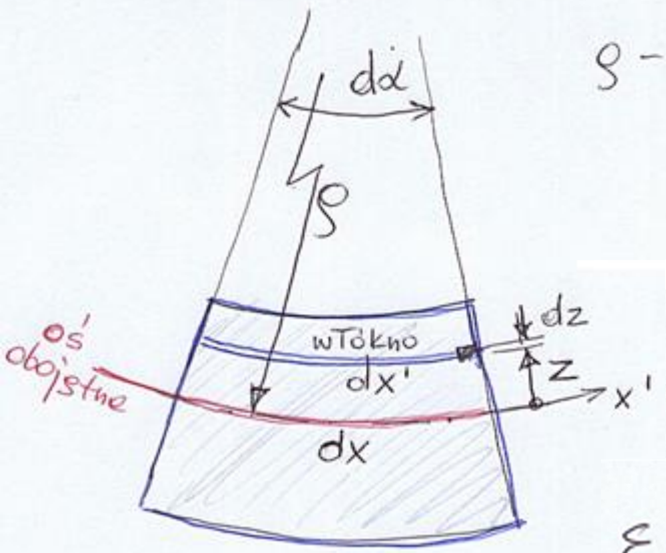
Po deformacji



Założenia:

1) przekroje płaskie, prostopadłe do osi belki przed odkształceniem muszą zostać płaskie i prostopadłe do zakrzywionej osi belki
2) Naciski (naprężenia) w kierunku poprzecznym do włókien można zaniedbać
3) Nie ma naprężeń tnących między włóknami

Stan deformacji



ρ – promień krzywizny

$$d\alpha = \frac{dx'}{\rho - z} = \frac{dx}{\rho}$$

$$\frac{dx'}{dx} = \frac{\rho - z}{\rho}$$

$$\varepsilon_x(z) = \frac{dx' - dx}{dx} = \frac{\rho - z}{\rho} - \frac{\rho}{\rho} \rightarrow \boxed{\varepsilon_x(z) = -\frac{z}{\rho}}$$

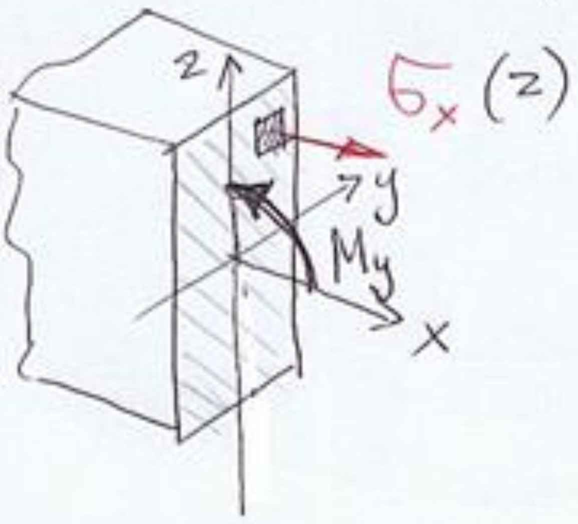
Prawo Hooke'a:

σ_x

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\underset{0}{\sigma_y} + \underset{0}{\sigma_z}\right)\right)$$

$$\boxed{\varepsilon_x = \frac{\sigma_x}{E}} \Rightarrow \boxed{\sigma_x = -\frac{E z}{\rho}}$$

Naprężenia normalne



1) $\underline{N=0}$ (brak rozciągania)

$$N = \int_A \sigma_x \, dA = -\frac{E}{\rho}\int_A z \, dA = 0$$

$$\boxed{S_y^c = 0}$$

oś y jest centralna!

2) $\underline{M_z=0}$ (brak zginania wzgl. osi z)

$$M_z = \int_A y \cdot \sigma_x \, dA = -\frac{E}{\rho}\int_A yz \, dA = 0$$

$$\boxed{J_{yz} = 0}$$

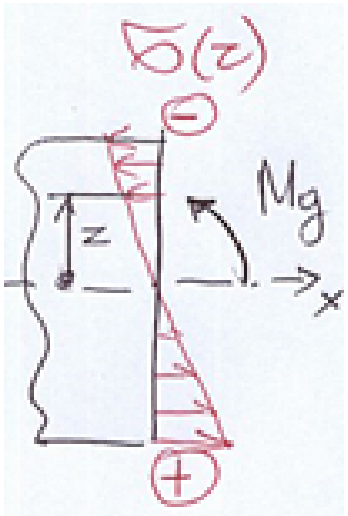
oś y jest główna!

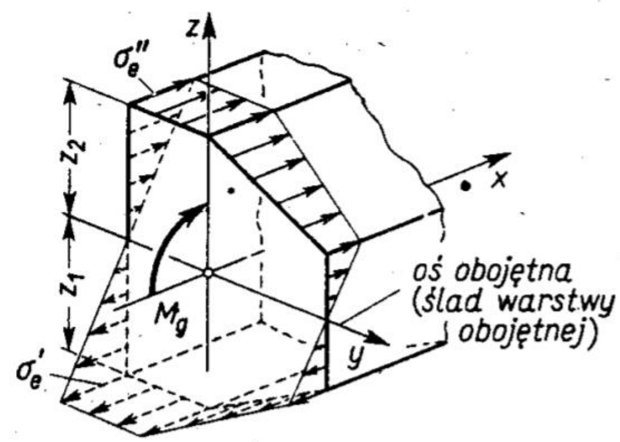
3) $\underline{M_y \neq 0}$

$$M_y = -\int_A z \, \sigma_x \, dA = +\frac{E}{\rho}\underbrace{\int_A z^2 dA}_{= J_y}$$

$$\boxed{\frac{1}{\rho} = \frac{M_y}{EJ_y}}$$

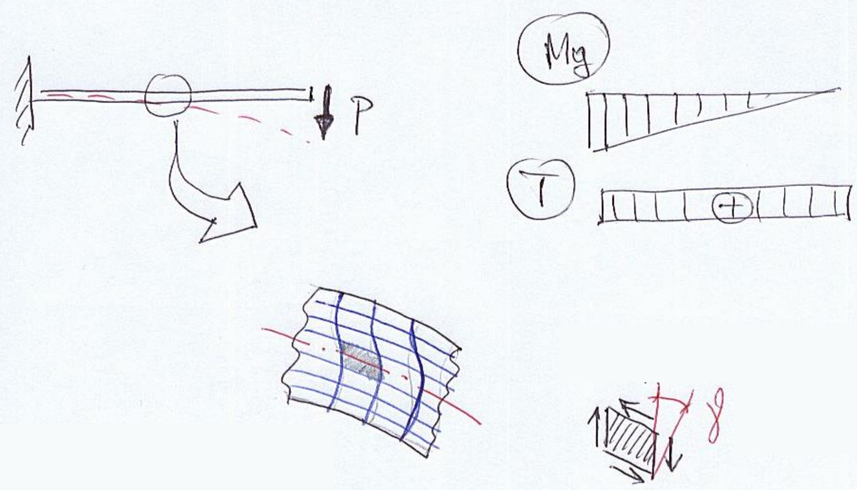
$$\sigma(z) = -\frac{Ez}{\rho} = \boxed{-\frac{M_y}{J_y}\cdot z}$$

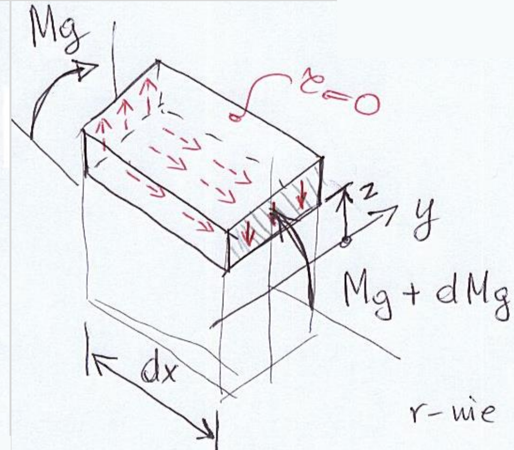


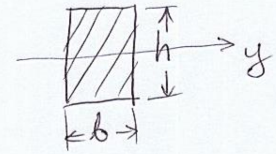


Naprężenia tnące

Zginanie poprzeczne $(T \neq 0)$







r-nie r-gi na kier x :

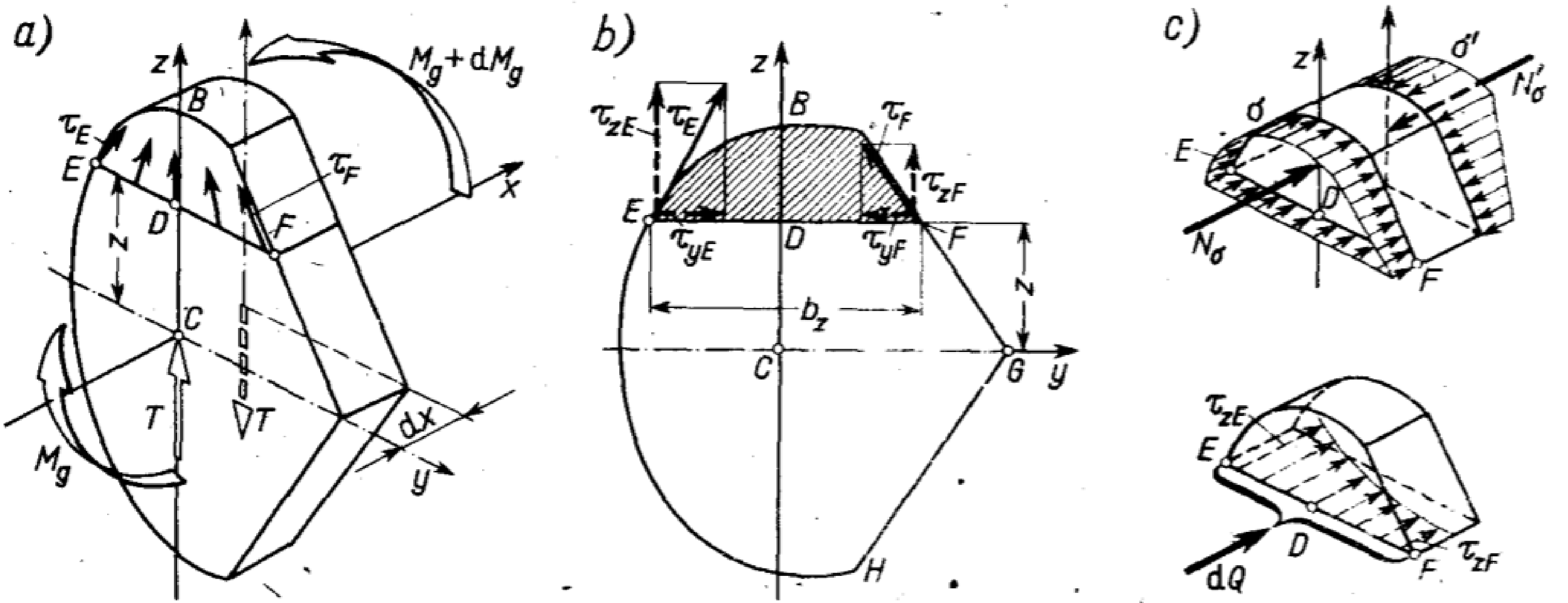
$$\underbrace{\int_z^{h/2} \sigma_x \, dA}_{\text{wyp. siła normalna}} + \tau \underbrace{b \cdot dx}_{\text{pole}} = 0$$

$$\int_z^{h/2} -\frac{dM_g}{J_y} \cdot z \cdot dA + \tau \, b \, dx = 0 \quad / :(dx \cdot b)$$

$$-\underbrace{\frac{dM_g}{dx}}_{T(x)} \frac{1}{b \cdot J_y} \underbrace{\int_z^{h/2} z \, dA}_{S_y \text{ (moment statyczny)}} + \tau = 0$$

$$\boxed{\tau = \frac{T(x) \cdot S_y(z)}{J_y \cdot b}}$$

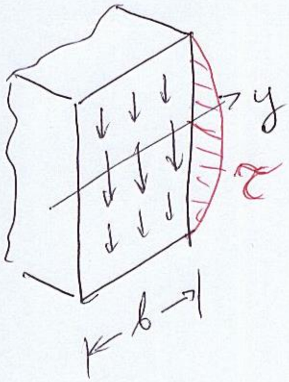
Naprężenia tnące



Rys. 5.39. Określenie naprężeń τ w belce o przekroju zwartym

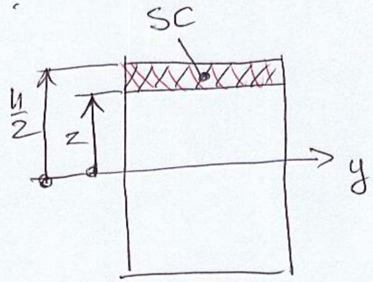
Przykład 1

dla przekroju prostokątnego $(b\times h)$



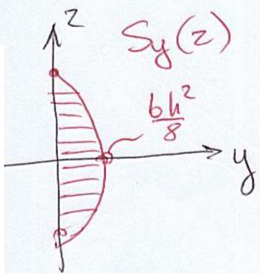
$$\boxed{\tau = \frac{T\cdot S_y}{J_y\cdot b}}\ ;$$

$$\boxed{J_y = \frac{b h^3}{12}}$$



$$S_y = \int_z^{h/2} z\,dA$$

$$S_y = \underbrace{\left(\frac{h}{2}+z\right)\cdot\frac{1}{2}}_{\text{wsp. SC}}\cdot\underbrace{\left(\frac{h}{2}-z\right)\cdot b}_{\text{pole}}$$

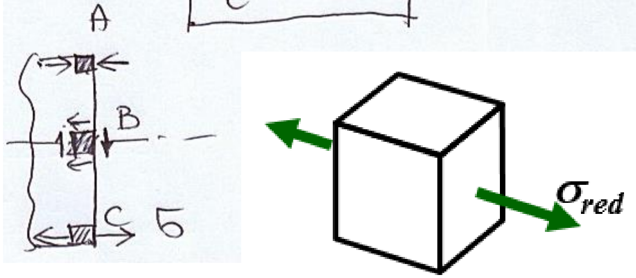


$$\boxed{S_y = \frac{1}{2}\left(\frac{h^2}{4}-z^2\right)\cdot b}$$

moment statyczny części przekroju

W belkach mamy zwykle

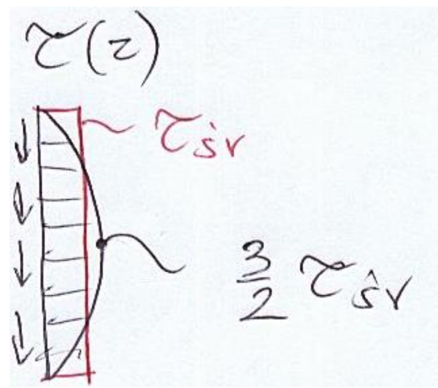
$$\boxed{\tau \ll \sigma}$$



$$S_y\left(z=\pm\frac{h}{2}\right)=0$$

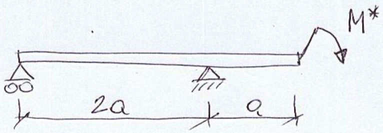
$$S_y(z=0) = \frac{b\cdot h}{2}\cdot\frac{h}{4} = \frac{bh^2}{8}$$

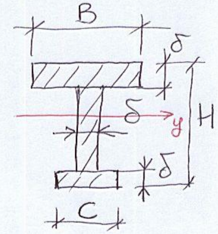
$$\tau_{max} = \tau(z=0) = \frac{T\cdot\frac{b\cdot h^2}{8}}{\frac{bh^3}{12}\cdot b} = \frac{3}{2}\frac{T}{b\cdot h} = \boxed{\frac{3}{2}\cdot\tau_{śr}}$$



Przykład 2

Zadanie





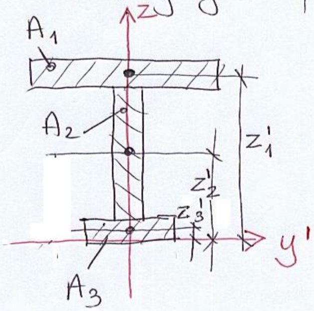
Dane: $M^* = 10$ kNm, $E = 2\cdot 10^5$ MPa

$a = 1$ m, $\boxed{B = 100,\ H = 120,\ C = 50,\ \delta = 10}$

Wyznaczyć: $Mg(x)$, $T(x)$, $w(x)$ – linię ugięcia

$(\sigma_{red})_{max} = ?$

Charakterystyki przekroju



– środek ciężkości przekroju

$A_1 = B\cdot\delta = 10\,cm \cdot 1\,cm = 10\,cm^2$

$z_1' = H - \delta/2 = 12 - 0.5 = 11.5\,cm$

$A_2 = (H-2\delta)\cdot\delta = 10\cdot 1 = 10\,cm^2$

$z_2' = H/2 = 6\,cm$

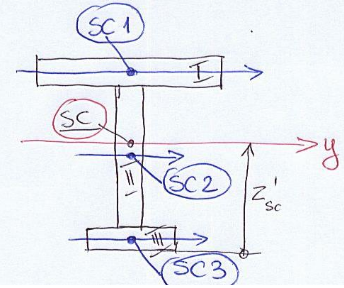
$A_3 = C\cdot\delta = 5\cdot 1 = 5\,cm^2$

$z_3' = \delta/2 = 0.5\,cm$

$$z'_{sc} = \frac{1}{A_1+A_2+A_3}\left(A_1\cdot z_1' + A_2\cdot z_2' + A_3\cdot z_3'\right)$$

$$z'_{sc} = \frac{1}{25\,cm^2}\left(10\cdot 11{,}5 + 10\cdot 6 + 5\cdot 0{,}5\right) = 7{,}1\,cm$$

własne centralne mom. bezw.



$$\boxed{J_1^{wł} = \frac{B\cdot\delta^3}{12}} = 0.833\,cm^4$$

$$\boxed{J_2^{wł} = \frac{\delta\cdot(H-2\delta)^3}{12}} = 83.3\,cm^4$$

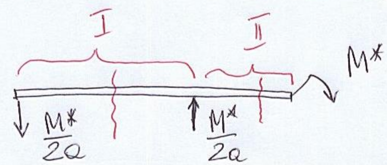
$$\boxed{J_3^{wł} = \frac{C\cdot\delta^3}{12}} = 0.417\,cm^4$$

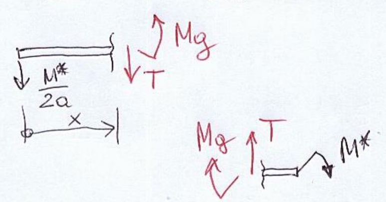
$$J_y \overset{t.S}{=} \boxed{J_1^{wł} + A_1\cdot(z_1' - z_{sc}')^2} + \boxed{J_2^{wł} + A_2(z_2' - z_{sc}')^2} + \boxed{J_3^{wł} + A_3\cdot(z_3' - z_{sc}')^2}$$

$$J_y = 0.833 + 10\cdot(11.5-7.1)^2 + 83.3 + 10\cdot(6-7.1)^2 + 0.417 + 5\cdot(0.5-7.1)^2 = 194.43 + 95.4 + 218.22$$

$\boxed{J_y = 508\ cm^4}$ moment bezwładności całego przekroju

II Wyznaczenie sił wewnętrznych





I $x\in(0, 2a)$

$$\boxed{T = -\frac{M^*}{2a}} = \frac{-10}{2\cdot 1} = -5\,(kN)$$

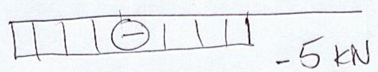
$$\boxed{Mg = -\frac{M^*}{2a}\cdot x} = -5x\,(kNm)$$

II $x\in(2a, 3a)$

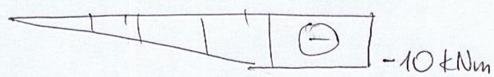
$\boxed{T = 0}$

$\boxed{Mg = -M^*} = -10\,(kNm)$

$T(x)$



$M_g(x)$



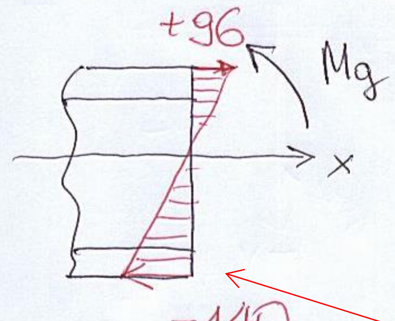
III Naprężenia normalne

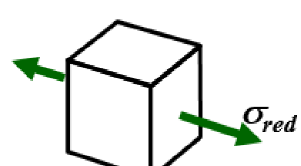
$$\sigma^G = -\frac{M_g(x)}{J_y}\cdot z_{max} = -\frac{(-10\cdot 10^3\ Nm)}{508\cdot 10^{-8}\ m^4}\cdot 4.9\cdot 10^{-2}\ m$$

$$\boxed{\sigma^G} = 96\cdot 10^6\ \frac{N}{m^2} = \boxed{96\ MPa}$$

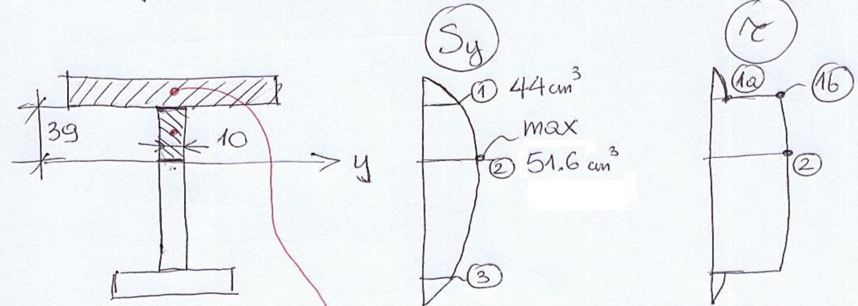
$$\sigma^D = -\frac{M_g(x)}{J_y}\cdot z_{min} = -\frac{(-10\cdot 10^3\ Nm)}{508\cdot 10^{-8}\ m^4}\left(-7.1\cdot 10^{-2}\ m\right)$$

$$\boxed{\sigma^D} = -140\cdot 10^6\ \frac{N}{m^2} = \boxed{-140\ MPa}$$





IV Naprężenia tnące



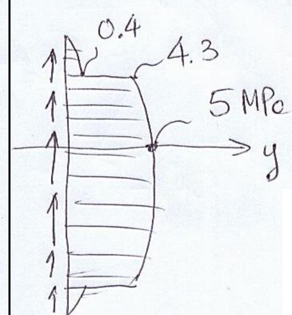
$$S_y^{①} = \boxed{B\cdot\delta}\cdot(z_1' - z_{sc}') = \int_{①}^{z_{max}} z\,dA = 44\ cm^3$$

$$S_y^{②} = \underbrace{B\cdot\delta\cdot(z_1' - z_{sc}')}_{44\ cm^3} + \boxed{3.9\cdot 1}\ \frac{3.9}{2} = 51.6\ cm^3$$

$$\tau^{1a} = \frac{T\cdot S_y^{①}}{J_y\cdot B} = \frac{-5\cdot 10^3\ N\cdot 44\ cm^3}{508\ cm^4\cdot 10\ cm} = -43\ \frac{N}{cm^2} = -0.43\ MPa$$

$$\tau^{1b} = \frac{T\cdot S_y^{①}}{J_y\cdot\delta} = \frac{-5\cdot 10^3\ N\cdot 44\ cm^3}{508\ cm^4\cdot 1\ cm} = -430\ \frac{N}{cm^2} = -4.3\ MPa$$

$$\tau^{②} = \frac{T\cdot S_y^{②}}{J_y\cdot\delta} = -\frac{5\cdot 10^3\ N\cdot 51.6\ cm^3}{508\ cm^4\cdot 1\ cm} = -508\ \frac{N}{cm^2} = -5\ MPa$$

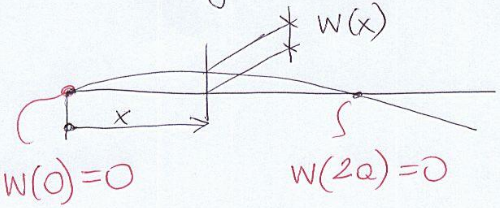


szacowanie napr. tnących

$$\tau_{śr} \cong \frac{T}{A_2} = \frac{5000\ N}{1000\ mm^2} = 5\ MPa$$

pole środnika

V Linia ugięcia



$$w'' \cong \frac{1}{\rho} = \frac{M_g(x)}{EJ_y}$$

$x \in (0, 2a)$

$$w_I'' = \frac{M_{gI}}{EJ_y} = \boxed{-\frac{M^* \cdot x}{2aEJ_y}}$$

$$w_I' = -\frac{M^* x^2}{4aEJ_y} + C_I$$

$$w_I = -\frac{M^* x^3}{12aEJ_y} + C_I x + D_I$$

$x \in (2a, 3a)$

$$w_{II}'' = \frac{M_{gII}}{EJ_y} = \boxed{-\frac{M^*}{EJ_y}}$$

$$w_{II}' = -\frac{M^* x}{EJ_y} + C_{II}$$

$$w_{II} = -\frac{M^* x^2}{2EJ_y} + C_{II} x + D_{II}$$

warunki brzegowe:

$$w_I(0) = D_I = 0$$

$$w_I(2a) = \frac{-M^*(2a)^3}{12aEJ_y} + C_I \cdot 2a = 0$$

warunki ciągłości:

$$w_I(2a) = w_{II}(2a) = -\frac{M^*(2a)^2}{2EJ_y} + C_{II} 2a + D_{II} = 0$$

$$w_I'(2a) = -\frac{M^* \cdot (2a)^2}{4aEJ_y} + C_I = w_{II}'(2a) = -\frac{M^* \cdot 2a}{EJ_y} + C_{II}$$

$$\boxed{C_I = \frac{M^* a}{3EJ}} = \frac{10 \cdot 10^3\ Nm \cdot 1m}{3 \cdot 2 \cdot 10^{11} \frac{N}{m^2} \cdot 508 \cdot 10^{-8} m^4} = 0.00328\,rad$$
$(0.19°)$

$$\boxed{D_I = 0}$$

$$\boxed{C_{II} = \frac{4}{3}\frac{M^* a}{EJ}} = 0.0131\ rad\ (0.752°)$$

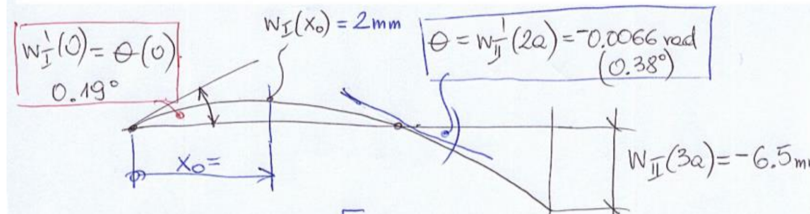
$$\boxed{D_{II} = -\frac{2}{3}\frac{M^* a^2}{EJ}} = -0.0066\ m = -6.6\,mm$$

$$w_I = \frac{M^*}{12aEJ}(4a^2 - x^2) \cdot x$$

$$w_I' = \frac{M^*}{12aEJ}(4a^2 - 3x^2)$$

$$w_{II} = \frac{M^*}{6EJ}(2a - x)(3x - 2a)$$

$$w_{II}' = \frac{M^*}{3EJ}(4a - 3x)$$



$$w_I'(x_0) = 0 \Rightarrow x_0 = \sqrt{\frac{4}{3}}\,a = 1.154\,m$$